On the maximum size of a (k, l)-sum-free subset of an abelian group

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Abstract

A subset A of a given finite abelian group G is called (k, l)-sum-free if the sum of k (not necessarily distinct) elements of A does not equal the sum of l (not necessarily distinct) elements of A. We are interested in finding the maximum size $\lambda_{k,l}(G)$ of a (k, l)-sum-free subset in G.

A $(2, 1)$-sum-free set is simply called a sum-free set. The maximum size of a sum-free set in the cyclic group $\mathbb{Z}_n$ was found almost forty years ago by Diamanda and Yap; the general case for arbitrary finite abelian groups was recently settled by Green and Ruzsa. Here we find the value of $\lambda_{3,1}(\mathbb{Z}_n)$. More generally, a recent paper of Hamidoune and Plagne examines (k, l)-sum-free sets in G when $k - l$ and the order of G are relatively prime; we extend their results to see what happens without this assumption.

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1 Introduction

Throughout this paper, we let G be a finite abelian group of order $n > 1$, written in additive notation; v will denote the exponent (i.e. largest order of any element) of G.

For subsets A and B of G, we use the standard notations $A + B$ and $A - B$ to denote the set of all two-term sums and differences, respectively, with one term chosen from A and one from B. If,

say, A consists of a single element a, then we simply write $a+B$ and $a-B$ instead of $A+B$ and $A-B$. For a positive integer h and a subset A of G, the set of all h-term sums with (not necessarily distinct) elements from A will be denoted by hA.

Let k and l be distinct positive integers. A subset A of G is called a (k,l)-*sum-free set* in G if

$$kA \cap lA = \emptyset;$$

or, equivalently, if

$$0 \notin kA - lA.$$

Clearly, we may assume that $k > l$. We are interested in determining the maximum possible size $\lambda_{k,l}(G)$ of a (k,l)-sum-free set in G.

A $(2,1)$-sum-free set is simply called a sum-free set. The value of $\lambda_{2,1}(\mathbb{Z}_n)$ was determined by Diamanda and Yap [13] in 1969. It can be proved (see also [31]) that

$$\max_{d|v}\left\{\left\lfloor\frac{d+1}{3}\right\rfloor\cdot\frac{n}{d}\right\} \le \lambda_{2,1}(G) \le \max_{d|n}\left\{\left\lfloor\frac{d+1}{3}\right\rfloor\cdot\frac{n}{d}\right\}, \tag{1}$$

which for cyclic groups immediately implies the following.

Theorem 1 (Diamanda and Yap [13]) *The maximum size $\lambda_{2,1}(\mathbb{Z}_n)$ of a sum-free set in the cyclic group of order n is given by*

$$\lambda_{2,1}(\mathbb{Z}_n) = \max_{d|n}\left\{\left\lfloor\frac{d+1}{3}\right\rfloor\cdot\frac{n}{d}\right\} = \begin{cases} \frac{p+1}{p}\cdot\frac{n}{3} & \text{if } n \text{ is divisible by a prime } p \equiv 2 \pmod 3 \\ & \text{and } p \text{ is the smallest such prime;} \\ \left\lfloor\frac{n}{3}\right\rfloor & \text{otherwise.} \end{cases}$$

The problem of finding $\lambda_{2,1}(G)$ for arbitrary G stood open for over 35 years. In a recent breakthrough paper, Green and Ruzsa [15] proved that, as it has been conjectured, the value of $\lambda_{2,1}(G)$ agrees with the lower bound in (1):

Theorem 2 (Green and Ruzsa [15]) *The maximum size $\lambda_{2,1}(G)$ of a sum-free set in G is*

$$\lambda_{2,1}(G) = \lambda_{2,1}(\mathbb{Z}_v)\cdot\frac{n}{v} = \max_{d|v}\left\{\left\lfloor\frac{d+1}{3}\right\rfloor\cdot\frac{n}{d}\right\}.$$

As a consequence, we see that

$$\frac{2}{7}n \le \lambda_{2,1}(G) \le \frac{1}{2}n$$

for every G, with equality holding in the lower bound when $v=7$ and in the upper bound when v (iff n) is even.

Now let us consider other values of k and l. In Section 2 of this paper we generalize (1), and prove the following.

Theorem 3 *The maximum size $\lambda_{k,l}(G)$ of a (k,l)-sum-free set in G satisfies*

$$\max_{d|v}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}\leq\lambda_{k,l}(G)\leq\max_{d|n}\left\{\left(\left\lfloor\frac{d-2}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\},$$

where $\delta(d)=\gcd(d,k-l)$.

Note that for $(k,l)=(2,1)$ Theorem 3 yields (1). Note also that, if $k-l$ is not divisible by v, then $\delta(v)=\gcd(v,k-l)\leq v/2$; in particular,

$$\lambda_{k,l}(G)\geq\frac{n}{2(k+l)}>0.$$

If, on the other hand, $k-l$ is divisible by v, then clearly $\lambda_{k,l}(G)=0$, since for any $a\in G$ we have $ka=la$.

Let us now consider cyclic groups. When $G\cong\mathbb{Z}_n$ and n and $k-l$ are relatively prime, then Theorem 3 gives

$$\lambda_{k,l}(\mathbb{Z}_n)=\max_{d|n}\left\{\left(\left\lfloor\frac{d-2}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}. \tag{2}$$

This result was already established by Hamidoune and Plagne in [17]. Their method was based on a generalization of Vosper's Theorem [30] on critical pairs where arithmetic progressions, that is, sets of the form

$$A=\{a,a+d,\ldots,a+c\cdot d\}$$

play a crucial role. In particular, Hamidoune and Plagne proved that, if $G\cong\mathbb{Z}_n$ and n and $k-l$ are relatively prime, then

$$\lambda_{k,l}(\mathbb{Z}_n)=\max_{d|n}\left\{\alpha_{k,l}(\mathbb{Z}_d)\cdot\frac{n}{d}\right\}, \tag{3}$$

where $\alpha_{k,l}(\mathbb{Z}_n)$ is the maximum size of a (k,l)-sum-free arithmetic progression in $\mathbb{Z}_n$. Hamidoune and Plagne deal only with the case when n and $k-l$ are relatively prime; as they point out, "in the absence of this assumption, degenerate behaviors may appear", and we concur with this assessment. Nevertheless, we attempt to treat the general case; in Section 3 of this paper we prove that (3) remains valid even without the assumption that n and $k-l$ are relatively prime:

Theorem 4 *For arbitrary positive integers k, l, and n we have*

$$\lambda_{k,l}(\mathbb{Z}_n)=\max_{d|n}\left\{\alpha_{k,l}(\mathbb{Z}_d)\cdot\frac{n}{d}\right\}.$$

Let us now move on to general abelian groups. Hamidoune and Plagne conjecture in [17] that

$$\lambda_{k,l}(G)=\lambda_{k,l}(\mathbb{Z}_v)\cdot\frac{n}{v}$$

holds when n and $k-l$ are relatively prime. They prove this assertion with the additional assumption that at least one prime divisor of v is not congruent to 1 (mod $k+l$). We generalize this result for the case when n and $k-l$ are not necessarily relatively prime:

Theorem 5 *As before, for a positive integer d, we set $\delta(d) = \gcd(d, k-l)$. If v possesses at least one divisor d which is not congruent to any integer between 1 and $\delta(d)$ (inclusive) (mod $k+l$), then*

$$\lambda_{k,l}(G) = \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}.$$

We closely follow some of the fundamental work of Hamidoune and Plagne in [17]; in fact, Section 3 of this paper can be considered an extention of [17] for the case when n and $k-l$ are not assumed to be relatively prime.

In Section 4 we employ Theorem 4 to establish the value of $\lambda_{3,1}(\mathbb{Z}_n)$ explicitly. As an analogue to Theorem 1 we prove the following.

Theorem 6 *The maximum size $\lambda_{3,1}(\mathbb{Z}_n)$ of a $(3,1)$-sum-free set in the cyclic group of order n is given by*

$$\lambda_{3,1}(\mathbb{Z}_n) = \max_{\substack{d|n \\ d \not\equiv 2 \pmod 4}} \left\{ \left\lfloor \frac{d+2}{4} \right\rfloor \cdot \frac{n}{d} \right\} = \begin{cases} \frac{p+1}{p} \cdot \frac{n}{4} & \text{if } n \text{ is divisible by a prime } p \equiv 3 \pmod 4 \\ & \text{and } p \text{ is the smallest such prime;} \\ \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

As a consequence, we see that

$$\frac{1}{5}n \leq \lambda_{3,1}(\mathbb{Z}_n) \leq \frac{1}{3}n,$$

with equality holding in the lower bound when $n \in \{5, 10\}$ and in the upper bound when n is divisible by 3.

In our final section, Section 5, we provide some further comments and discuss several open questions about (k,l)-sum-free sets.

2 Bounds for the size of maximum (k,l)-sum-free sets

In this section we prove Theorem 3.

We will use the following easy lemma.

Lemma 7 *Suppose that A is a maximal (k,l)-sum-free set in G. Let K denote the stabilizer subgroup of kA. Then*

(i) $k(A+K) = kA$;

(ii) $A+K$ is a (k,l)-sum-free set in G;

(iii) $A+K = A$;

(iv) A is the union of cosets of K.

Proof. (i) The inclusion $kA \subseteq k(A+K)$ is obvious. Suppose that $a_1,\dots,a_k \in A$ and $h_1,\dots,h_k \in K$. Then
$$(a_1+\cdots+a_k)+(h_1+\cdots+h_k) \in kA,$$
so $k(A+K) \subseteq kA$.

(ii) Suppose, indirectly, that
$$k(A+K)\cap l(A+K) \neq \emptyset;$$
by (i) this implies
$$kA \cap l(A+K) \neq \emptyset.$$
Then we can find elements $a_1,\dots,a_k \in A$, $a'_1,\dots,a'_l \in A$, and $h_1,\dots,h_l \in K$ for which
$$a_1+\cdots+a_k = a'_1+\cdots+a'_l+h_1+\cdots+h_l.$$
But
$$a'_1+\cdots+a'_l = a_1+\cdots+a_k-h_1-\cdots-h_l \in kA,$$
and this contradicts the fact that A is (k,l)-sum-free.

(iii) Since $A \subseteq A+K$ and A is a maximal (k,l)-sum-free set in G, by (ii) we have $A+K=A$.

(iv) We need to show that for any $a \in A$, we have $a+K \subseteq A$. But $a+K \subseteq A+K$, so the claim follows from (iii). $\square$

For the upper bound in Theorem 3, we need the following result which is essentially due to Kneser.

Theorem 8 (Kneser [20]; see Theorem 4.4 in [25]) *Suppose that A is a non-empty subset of G and, for a given positive integer h, let H be the stabilizer of hA. Then we have*
$$|hA| \geq h\cdot|A|-(h-1)\cdot|H|.$$

Proof of the upper bound in Theorem 3. Let A be a (k,l)-sum-free set in G with $|A|=\lambda$; then we have
$$kA\cap lA=\emptyset$$
and therefore
$$n \geq |kA|+|lA|. \tag{4}$$
As before, let K and L be the stabilizer subgroups of kA and lA, respectively. Then, by Theorem 8, we have
$$|kA| \geq k\cdot|A|-(k-1)\cdot|K|$$
and
$$|lA| \geq l\cdot|A|-(l-1)\cdot|L|;$$
thus, from (4) we get
$$n \geq (k+l)\cdot|A|-(k-1)\cdot|K|-(l-1)\cdot|L|.$$
Without loss of generality we can assume that $|K|\geq|L|$, so
$$n \geq (k+l)\cdot|A|-(k+l-2)\cdot|K|$$

or

$$\frac{|A|}{|K|} \leq \frac{1}{k+l} \cdot \left(\frac{n}{|K|} + (k+l-2) \right).$$

Now $|A| = \lambda$; in particular, A is maximal, so by Lemma 7 (iv), $\frac{|A|}{|K|}$ must be an integer. Therefore, with d denoting the index of K in G, we get

$$\frac{\lambda}{n/d} \leq \left\lfloor \frac{1}{k+l} \cdot (d+k+l-2) \right\rfloor,$$

from which our claim follows. □

Proposition 9 *Let d be a positive integer, and set $\delta(d) = \gcd(d, k-l)$. Suppose that c is a positive integer for which*

$$(k+l) \cdot c \leq d - 1 - \delta(d).$$

Then there exists an element $a \in \mathbb{Z}_d$ for which the set

$$A = \{a, a+1, a+2, \ldots, a+c\}$$

is a (k,l)-sum-free in $\mathbb{Z}_d$ of size $c+1$.

Proof. By the Euclidean Algorithm, we have unique integers q and r for which

$$l \cdot c = \delta(d) \cdot q - r$$

and $1 \leq r \leq \delta(d)$. We also know the existence of integers u and v for which

$$\delta(d) = (k-l) \cdot u + d \cdot v.$$

Now set $a = u \cdot q$. We will show that

$$A = \{a, a+1, a+2, \ldots, a+c\}$$

is a (k,l)-sum-free in $\mathbb{Z}_d$. (Here, and elsewhere, we consider integers as elements of $\mathbb{Z}_d$ via the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}_d$.)

First note that, for any integer i with $-l \cdot c \leq i \leq k \cdot c$, our assumption about c implies

$$1 \leq r \leq l \cdot c + i + r \leq (k+l) \cdot c + r \leq (k+l) \cdot c + \delta(d) \leq d - 1,$$

and therefore, considering

$$B = \{l \cdot c + i + r \mid -l \cdot c \leq i \leq k \cdot c\}$$

as a subset of $\mathbb{Z}_d$, we have $0 \notin B$.

Furthermore, in $\mathbb{Z}_d$ we have

$$(k-l) \cdot a = (k-l) \cdot u \cdot q = \delta(d) \cdot q - d \cdot v \cdot q = \delta(d) \cdot q = l \cdot c + r,$$

and therefore

$$kA - lA = \{(k-l) \cdot a + i \mid -l \cdot c \leq i \leq k \cdot c\} = B.$$

Since $0 \notin B$, A is indeed (k,l)-sum-free in $\mathbb{Z}_d$.

Furthermore, since $c < d$, we see that $|A| = c+1$, as claimed. □

The lower bound in Theorem 3 now follows from Proposition 9 and the following lemma.

Lemma 10 *Suppose that d is a divisor of v. Then*

$$\lambda_{k,l}(G) \geq \lambda_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d}.$$

Proof. Since d is a divisor of v, there is a subgroup H of G of index d for which

$$G/H \cong \mathbb{Z}_d.$$

Let $\Phi : G \to G/H$ be the canonical homomorphism from G to G/H, and let $\Psi : G/H \to \mathbb{Z}_d$ be the isomorphism from G/H to $\mathbb{Z}_d$. Then, for any (k,l)-sum-free set $A \subseteq \mathbb{Z}_d$, the set $\Phi^{-1}(\Psi^{-1}(A))$ is a (k,l)-sum-free set in G and has size $\frac{n}{d} \cdot |A|$. $\square$

3 (k,l)-sum-free sets in cyclic groups

In this section we analyze (k,l)-sum-free arithmetic progressions in $\mathbb{Z}_n$ and prove Theorems 4 and 5. This was carried out by Hamidoune and Plagne in [17] with the assumption that n and $k-l$ are relatively prime; here we drop that assumption but follow their approach.

A subset A of $\mathbb{Z}_n$ is an arithmetic progression of difference $d \in \mathbb{Z}_n$, if

$$A = \{a, a+d, \ldots, a+c \cdot d\}$$

for some $a \in \mathbb{Z}_n$ and non-negative integer c. We let $A_{k,l}(n)$ be the set of (k,l)-sum-free arithmetic progression in $\mathbb{Z}_n$. We also let $B_{k,l}(n)$ and $C_{k,l}(n)$ be the sets of those sequences in $A_{k,l}(n)$ whose difference is not relatively prime to n, and relatively prime to n, respectively. Note that a sequence can belong to both $B_{k,l}(n)$ and $C_{k,l}(n)$ only if it contains exactly 1 term, and that sequences in $B_{k,l}(n)$ are each contained in a proper coset in $\mathbb{Z}_n$, while no sequence in $C_{k,l}(n)$ with more than one term is contained in a proper coset.

We introduce the following notations.

$$\alpha_{k,l}(\mathbb{Z}_n) = \max\{|A| \mid A \in A_{k,l}(n)\}$$

$$\beta_{k,l}(\mathbb{Z}_n) = \max\{|A| \mid A \in B_{k,l}(n)\}$$

$$\gamma_{k,l}(\mathbb{Z}_n) = \max\{|A| \mid A \in C_{k,l}(n)\}$$

Clearly, $\alpha_{k,l}(\mathbb{Z}_n) = \max\{\beta_{k,l}(\mathbb{Z}_n), \gamma_{k,l}(\mathbb{Z}_n)\}$.

We also let $D(n)$ be the set of all divisors of n which are greater than 1. Furthermore, we separate the elements of $D(n)$ into subsets $D_1(n)$ and $D_2(n)$ according to whether they do not or do divide $k-l$, respectively. Then the following are clear:

- $D_1(n) = \emptyset$ if, and only if, $k-l$ is divisible by n;
- $D_2(n) = \emptyset$ if, and only if, $k-l$ and n are relatively prime; and
- $D_1(n) \neq \emptyset$ and $D_2(n) \neq \emptyset$ if, and only if, $1 < \gcd(n, k-l) < n$.

The next three propositions summarize our results on $\alpha_{k,l}(\mathbb{Z}_n)$, $\beta_{k,l}(\mathbb{Z}_n)$, and $\gamma_{k,l}(\mathbb{Z}_n)$. We start with $\beta_{k,l}(\mathbb{Z}_n)$.

Proposition 11 *The maximum size $\beta_{k,l}(\mathbb{Z}_n)$ of a (k,l)-sum-free arithmetic progression in $\mathbb{Z}_n$ whose difference is not relatively prime to n satisfies the following.*

(i) If $k-l$ is divisible by n, then $\beta_{k,l}(\mathbb{Z}_n)=0$.

(ii) If $k-l$ and n are relatively prime, then $\beta_{k,l}(\mathbb{Z}_n)=\frac{n}{p}$ where p is the smallest prime divisor of n.

(iii) If $1<\gcd(n,k-l)<n$, then we have

$$\frac{n}{\rho_1} \leq \beta_{k,l}(\mathbb{Z}_n) \leq \max\left\{\frac{n}{\rho_1}, \frac{n}{2\rho_2}\right\},$$

where ρ_1 and ρ_2 are the smallest elements of $D_1(n)$ and $D_2(n)$, respectively.

Proof. If n divides $k-l$, then for any $a \in \mathbb{Z}_n$ we have $ka=la$. This implies (i). Statements (ii) and (iii) will follow from the following three claims.

Claim 1. Suppose that $d \in D_1(n)$. Then the set

$$A=\left\{1+i\cdot d \mid 0 \leq i \leq \frac{n}{d}-1\right\}$$

is an arithmetic progression in $B_{k,l}(n)$, has size $|A|=\frac{n}{d}$, and is (k,l)-sum-free.

Proof of Claim 1. Clearly, A belongs to $B_{k,l}(n)$ and has size $|A|=\frac{n}{d}$. Furthermore,

$$kA-lA=\left\{(k-l)+d\cdot j \mid \ -l\cdot\left(\frac{n}{d}-1\right) \leq j \leq k\cdot\left(\frac{n}{d}-1\right)\right\}.$$

Since $d|n$ but $d \not| (k-l)$, we have $0 \not\in kA-lA$ which means that A is (k,l)-sum-free.

Claim 2. Suppose that H is a subgroup of $\mathbb{Z}_n$ of index d, and that A is a (k,l)-sum-free subset of $\mathbb{Z}_n$ (not necessarily an arithmetic progression) which lies in a single coset of H. Then $|A| \leq \frac{n}{d}$.

Proof of Claim 2. Clearly, $A \subseteq a+H$ implies $|A| \leq |H| = \frac{n}{d}$.

Claim 3. Suppose again that H is a subgroup of $\mathbb{Z}_n$ of index d, and that A is a (k,l)-sum-free subset of $\mathbb{Z}_n$ which lies in a single coset of H. If $d \in D_2(n)$, then $|A| \leq \frac{n}{2d}$.

Proof of Claim 3. Note that H is a cyclic group of order n/d and

$$H=\left\{0,d,2d,\ldots,\frac{n}{d}-1\right\}.$$

Since A lies in a single coset of H, so do kA and lA. But $k-l$ is divisible by d, so $ka-la \in H$, and therefore the sets kA and lA lie in the same coset of H. Thus we have

$$|kA \cup lA| \leq |H| = \frac{n}{d}.$$

But A is (k,l)-sum-free, so kA and lA must be disjoint, hence

$$|kA|+|lA| \leq \frac{n}{d}.$$

Now clearly $(k-1)a + A \subseteq kA$, so $|A| \leq |kA|$; similarly, $|A| \leq |lA|$. This implies that

$$|A| + |A| \leq \frac{n}{d}.$$

□

Next, we turn to $\gamma_{k,l}(\mathbb{Z}_n)$.

Proposition 12 *The maximum size $\gamma_{k,l}(\mathbb{Z}_n)$ of a (k,l)-sum-free arithmetic progression in $\mathbb{Z}_n$ whose difference is relatively prime to n satisfies*

$$\left\lfloor \frac{n-1-\delta}{k+l} \right\rfloor + 1 \leq \gamma_{k,l}(\mathbb{Z}_n) \leq \left\lfloor \frac{n-2}{k+l} \right\rfloor + 1,$$

where $\delta = \gcd(n, k-l)$.

Proof. The lower bound follows directly from Proposition 9.

For the upper bound, suppose that $d \in \mathbb{Z}_n$ and $\gcd(d,n) = 1$, and let $a \in \mathbb{Z}_n$. We need to show that, if the set

$$A = \{a, a+d, \ldots, a + c \cdot d\}$$

is (k,l)-sum-free in $\mathbb{Z}_n$, then

$$(k+l) \cdot c \leq n-2.$$

Suppose, indirectly, that

$$(k+l) \cdot c \geq n-1;$$

then we have

$$\{(k-l) \cdot a + i \cdot d \mid -l \cdot c \leq i \leq k \cdot c\} \supseteq \{(k-l) \cdot a + j \cdot d \mid 0 \leq j \leq n-1\}.$$

Now the left-hand side equals $kA - lA$. Since $\gcd(d,n) = 1$, the right-hand side equals the entire group $\mathbb{Z}_n$. But then $kA - lA$ must contain 0, which is a contradiction. □

We can now combine Propositions 11 and 12 to get results for the maximum size of (k,l)-sum-free arithmetic progressions in $\mathbb{Z}_n$.

Proposition 13 *The maximum size $\alpha_{k,l}(\mathbb{Z}_n)$ of a (k,l)-sum-free arithmetic progression in $\mathbb{Z}_n$ satisfies the following.*

(i) If $k-l$ is divisible by n, then $\alpha_{k,l}(\mathbb{Z}_n) = 0$.

(ii) If $k-l$ and n are relatively prime, then

$$\alpha_{k,l}(\mathbb{Z}_n) = \max\left\{\frac{n}{p}, \left\lfloor \frac{n-2}{k+l} \right\rfloor + 1\right\}$$

where p is the smallest prime divisor of n.

(iii) If $1 < \gcd(n, k-l) < n$, then we have

$$\max\left\{\frac{n}{\rho_1}, \left\lfloor \frac{n-1-\delta}{k+l} \right\rfloor + 1\right\} \leq \alpha_{k,l}(\mathbb{Z}_n) \leq \max\left\{\frac{n}{\rho_1}, \frac{n}{2\rho_2}, \left\lfloor \frac{n-2}{k+l} \right\rfloor + 1\right\},$$

where $\delta = \gcd(n, k-l)$, and ρ_1 and ρ_2 are the smallest elements of $D_1(n)$ and $D_2(n)$, respectively.

It is easy to see that the bounds in Proposition 13 are tight.

Now we are ready to prove Theorem 4. Due to the following result in [17], our task is not difficult.

Theorem 14 (Hamidoune and Plagne, [17]) *Let ϵ be 0 if n is even and 1 if n is odd. Then we have the following bounds.*

$$\max_{d|v}\left\{\alpha_{k,l}(\mathbb{Z}_d)\cdot\frac{n}{d}\right\} \le \lambda_{k,l}(G) \le \max\left\{\frac{n-\epsilon}{k+l}, \max_{d|v}\left\{\alpha_{k,l}(\mathbb{Z}_d)\cdot\frac{n}{d}\right\}\right\}$$

Proof of Theorem 4. If $k-l$ is divisible by n, Theorem 4 obviously holds as both sides equal zero, so let's assume otherwise. By Theorem 14, it suffices to prove that

$$\left\lfloor\frac{n-\epsilon}{k+l}\right\rfloor \le \max_{d|n}\left\{\alpha_{k,l}(\mathbb{Z}_d)\cdot\frac{n}{d}\right\}.$$

By Proposition 13, this statement follows once we prove

$$\left\lfloor\frac{n-\epsilon}{k+l}\right\rfloor \le \max_{d|n}\left\{\max\left\{\frac{d}{\rho_1(d)}, \left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right\}\cdot\frac{n}{d}\right\}, \tag{5}$$

where $\rho_1(d)$ is the smallest divisor of d which does not divide $k-l$. (Note that in the case when $\delta = 1$, $\rho_1(d)$ is simply the smallest prime dividing d, thus we do not need to consider cases (ii) and (iii) of Proposition 13 separately.)

Now $\rho_1 = \rho_1(n)$ does not divide $k-l$, so we must have $\delta(\rho_1) = \gcd(\rho_1, k-l) < \rho_1$. Therefore, since ρ_1 divides n, we have

$$\max_{d|n}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\} \ge \left(\left\lfloor\frac{\rho_1-1-\delta(\rho_1)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{\rho_1} \ge \frac{n}{\rho_1}.$$

We then have

$$\begin{aligned}
&\max_{d|n}\left\{\max\left\{\frac{d}{\rho_1(d)}, \left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right\}\cdot\frac{n}{d}\right\} = \\
&\qquad = \max\left\{\max_{d|n}\left\{\frac{n}{\rho_1(d)}\right\}, \max_{d|n}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}\right\} \\
&\qquad = \max\left\{\frac{n}{\rho_1}, \max_{d|n}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}\right\} \\
&\qquad = \max_{d|n}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}.
\end{aligned}$$

Therefore, (5) is equivalent to

$$\left\lfloor\frac{n-\epsilon}{k+l}\right\rfloor \le \max_{d|n}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}.$$

But this inequality clearly holds, since

$$\begin{aligned}\max_{d|n}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\} &\geq \left\lfloor\frac{n-1-\delta}{k+l}\right\rfloor+1\\ &\geq \left\lfloor\frac{n-1-(k-l)}{k+l}\right\rfloor+1\\ &= \left\lfloor\frac{n+(2l-1)}{k+l}\right\rfloor\\ &\geq \left\lfloor\frac{n-\epsilon}{k+l}\right\rfloor.\end{aligned}$$

□

Proof of Theorem 5. By Theorems 4 and 14, here we need to show that our assumptions imply

$$\left\lfloor\frac{n-\epsilon}{k+l}\right\rfloor \leq \max_{d|v}\left\{\max\left\{\frac{d}{\rho_1(d)},\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right\}\cdot\frac{n}{d}\right\}, \tag{6}$$

where $\rho_1(d)$ is the smallest divisor of d which does not divide $k-l$. (The only difference between (5) and (6) is that in (6) only divisors of v are considered.)

In a similar manner as before, we use the fact that $\rho_1(v)$ does not divide $k-l$ to conclude that the right hand side equals

$$\max_{d|v}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\}.$$

Now let d_0 be a divisor of v which is not congruent to any integer between 1 and $\delta(d_0)$ (inclusive) (mod $k+l$). Then the remainder of $d_0-1-\delta(d_0)$ when divided by $k+l$ is at most $k+l-1-\delta(d_0)$. Therefore, we have

$$\begin{aligned}\max_{d|v}\left\{\left(\left\lfloor\frac{d-1-\delta(d)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d}\right\} &\geq \left(\left\lfloor\frac{d_0-1-\delta(d_0)}{k+l}\right\rfloor+1\right)\cdot\frac{n}{d_0}\\ &\geq \left(\frac{d_0-(k+l)}{k+l}+1\right)\cdot\frac{n}{d_0}\\ &= \frac{n}{k+l},\end{aligned}$$

proving (6). □

4 $(3,1)$-sum-free sets in cyclic groups

In this section we prove Theorem 6 and find $\lambda_{3,1}(\mathbb{Z}_n)$ explicitly. First, we evaluate $\alpha_{3,1}(\mathbb{Z}_n)$. We note that, while Proposition 13 (ii) readily yields

$$\alpha_{2,1}(\mathbb{Z}_n)=\begin{cases}\frac{n}{2} & \text{if} \quad 2|n;\\ \left\lfloor\frac{n+1}{3}\right\rfloor & \text{if} \quad 2\not|n;\end{cases}$$

evaluating $\alpha_{3,1}(\mathbb{Z}_n)$ requires a bit more work.

Proposition 15 *The maximum size $\alpha_{3,1}(\mathbb{Z}_n)$ of a $(3,1)$-sum-free arithmetic progression in $\mathbb{Z}_n$ is given as follows:*

$$\alpha_{3,1}(\mathbb{Z}_n) = \begin{cases} \frac{n}{3} & \text{if} \quad 3|n; \\ \left\lfloor \frac{n+2}{4} \right\rfloor & \text{if} \quad 3 \not| n \text{ and } n \not\equiv 2 \pmod 8; \\ \frac{n-2}{4} & \text{if} \quad 3 \not| n \text{ and } n \equiv 2 \pmod 8. \end{cases}$$

Proof. Let $\alpha_{3,1}(n) = \alpha$. If $n = 2$, the claim holds, so we assume that $n \geq 3$. We distinguish several cases.

Case 1: $2 \not| n$ and $3 \not| n$. In this case Proposition 13 (ii) applies, and

$$\alpha = \left\lfloor \frac{n+2}{4} \right\rfloor.$$

Case 2: $2 \not| n$ and $3|n$. Proposition 13 (ii) applies again; we get

$$\alpha = \max\left\{\frac{n}{3}, \left\lfloor \frac{n+2}{4} \right\rfloor\right\} = \frac{n}{3}.$$

Case 3: $2|n$ and $3|n$. In this case Proposition 13 (iii) applies with $\delta = 2$, $\rho_1 = 3$, and $\rho_2 = 2$; we get

$$\max\left\{\frac{n}{3}, \left\lfloor \frac{n+1}{4} \right\rfloor\right\} \leq \alpha \leq \max\left\{\frac{n}{3}, \frac{n}{4}, \left\lfloor \frac{n+2}{4} \right\rfloor\right\},$$

which again implies

$$\alpha = \frac{n}{3}.$$

Case 4: $4|n$ and $3 \not| n$. Again Proposition 13 (iii) applies – this time with $\delta = 2$, $\rho_1 = 4$, and $\rho_2 = 2$. Therefore we get

$$\max\left\{\frac{n}{4}, \left\lfloor \frac{n+1}{4} \right\rfloor\right\} \leq \alpha \leq \max\left\{\frac{n}{4}, \left\lfloor \frac{n+2}{4} \right\rfloor\right\},$$

which gives

$$\alpha = \frac{n}{4}.$$

Case 5: $n \equiv 2 \pmod 4$ and $3 \not| n$. Again Proposition 13 (iii) applies — this time with $\delta = 2$, $\rho_1 \geq 5$, and $\rho_2 = 2$. Therefore we get

$$\max\left\{\frac{n}{\rho_1}, \left\lfloor \frac{n+1}{4} \right\rfloor\right\} \leq \alpha \leq \max\left\{\frac{n}{\rho_1}, \frac{n}{4}, \left\lfloor \frac{n+2}{4} \right\rfloor\right\},$$

which yields only

$$\alpha \in \left\{\frac{n-2}{4}, \frac{n+2}{4}\right\}.$$

To continue further, we separate the cases of $n \equiv 2 \pmod 8$ and $n \equiv 6 \pmod 8$.

Case 5.1. Let us first consider the case when $n \equiv 6 \pmod 8$. With $a = \frac{n+2}{8}$ and $c = \frac{n-2}{4}$, we let

$$A = \{a, a+1, \ldots, a+c\}.$$

Then

$$3A - A = \{2a - c + i \mid 0 \leq i \leq 4c\} = \{1 + i \mid 0 \leq i \leq n-2\} = \mathbb{Z}_n \setminus \{0\},$$

so A is (3,1)-sum-free in $\mathbb{Z}_n$ of size $c + 1 = \frac{n+2}{4}$.

Case 5.2. Now suppose that $n \equiv 2 \pmod 8$. We prove that $\alpha = \frac{n-2}{4}$. Suppose, indirectly, that $\alpha = \frac{n+2}{4}$ and there is a (3,1)-sum-free arithmetic progression

$$A = \{a, a+d, \ldots, a + c \cdot d\}$$

in $\mathbb{Z}_n$ of size $c + 1 = \frac{n+2}{4}$. Similarly to above,

$$3A - A = \{2a - c \cdot d + i \cdot d \mid 0 \leq i \leq 4c\} = \{2a - c \cdot d + i \cdot d \mid 0 \leq i \leq n-2\}.$$

By Proposition 11 (iii), we have

$$\beta_{3,1}(n) \leq \max\left\{\frac{n}{\rho_1}, \frac{n}{4}\right\} = \frac{n}{4};$$

so we have $\beta_{3,1}(n) < \alpha$. Therefore, we must have $\gcd(d, n) = 1$, which implies that

$$|3A - A| = n - 1.$$

Since A is (3,1)-sum-free, $0 \notin 3A - A$, and this can only occur if

$$2a - c \cdot d + (n-1) \cdot d \equiv 0 \pmod n.$$

A simple parity argument provides a contradiction: $2a - c \cdot d + (n-1) \cdot d$ is odd, so it cannot be divisible by n. $\square$

Proof of Theorem 6. As previously, we let $D(n)$ be the set of divisors of n which are greater than 1. We introduce the following six (potentially empty) subsets of $D(n)$, as well as some notations.

$$\begin{array}{lllllll}
E_1(n) & = & \{d \in D(n) \mid 3|d\} & \quad & e_1 & = & \max_{d \in E_1(n)} \left\{\frac{d}{3} \cdot \frac{n}{d}\right\} \\
E_2(n) & = & \{d \in D(n) \mid d \equiv 3(4), 3 \nmid d\} & & e_2 & = & \max_{d \in E_2(n)} \left\{\frac{d+1}{4} \cdot \frac{n}{d}\right\} \\
E_3(n) & = & \{d \in D(n) \mid 4|d, 3 \nmid d\} & & e_3 & = & \max_{d \in E_3(n)} \left\{\frac{d}{4} \cdot \frac{n}{d}\right\} \\
E_4(n) & = & \{d \in D(n) \mid d \equiv 1(4), 3 \nmid d\} & & e_4 & = & \max_{d \in E_4(n)} \left\{\frac{d-1}{4} \cdot \frac{n}{d}\right\} \\
E_5(n) & = & \{d \in D(n) \mid d \equiv 6(8), 3 \nmid d\} & & e_5 & = & \max_{d \in E_5(n)} \left\{\frac{d+2}{4} \cdot \frac{n}{d}\right\} \\
E_6(n) & = & \{d \in D(n) \mid d \equiv 2(8), 3 \nmid d\} & & e_6 & = & \max_{d \in E_6(n)} \left\{\frac{d-2}{4} \cdot \frac{n}{d}\right\}
\end{array}$$

(We have the understanding that $\max \emptyset = 0$.)

Then we have

$$D(n) = \cup_{i=1}^{6} E_i(n);$$

furthermore, by Theorem 4 and Proposition 15, we have

$$\lambda_{3,1}(\mathbb{Z}_n) = \max\{e_i | 1 \leq i \leq 6\}.$$

For any $i \in \{1, 2, \ldots, 6\}$ for which $E_i(n) \neq \emptyset$, we let

$$p_i = \min\{E_i(n)\}$$

and

$$n_i = \max\{E_i(n)\}.$$

Now suppose that $E_5(n) \neq \emptyset$. Then $E_2(n) \neq \emptyset$, and $p_5 = 2 \cdot p_2$. Therefore

$$e_5 = \frac{p_5 + 2}{4} \cdot \frac{n}{p_5} = \frac{p_2 + 1}{4} \cdot \frac{n}{p_2} = e_2.$$

We can similarly show that, if $E_6(n) \neq \emptyset$, then $E_4(n) \neq \emptyset$ and $e_6 = e_4$. Therefore, we see that

$$\lambda_{3,1}(\mathbb{Z}_n) = \max\{e_i | 1 \leq i \leq 4\}.$$

Next, observe that, if $E_i(n) \neq \emptyset$ for some $i \in \{1, 2, 3\}$, then $e_i \geq e_j$ for all $i < j \leq 4$.

Now we consider the following cases.

Case 1. Suppose that n has divisors which are congruent to 3 mod 4, and let p be the smallest such divisor. If $p = 3$, then $E_1(n) \neq \emptyset$, thus

$$\lambda_{3,1}(\mathbb{Z}_n) = e_1 = \frac{n}{3}.$$

If, on the other hand, $p > 3$, then $E_1(n) = \emptyset$ but $E_2(n) \neq \emptyset$, thus

$$\lambda_{3,1}(\mathbb{Z}_n) = e_2 = \frac{p+1}{p} \cdot \frac{n}{4}.$$

Case 2. Suppose that n has no divisors which are congruent to 3 mod 4, but that n is divisible by 4. In this case, $E_1(n) = E_2(n) = \emptyset$ but $E_3(n) \neq \emptyset$, thus

$$\lambda_{3,1}(\mathbb{Z}_n) = e_3 = \frac{n}{4}.$$

Case 3. Suppose that n has no divisors which are congruent to 3 mod 4, and that n is not divisible by 4. In this case, $E_1(n) = E_2(n) = E_3(n) = \emptyset$ but $E_4(n) \neq \emptyset$, thus

$$\lambda_{3,1}(\mathbb{Z}_n) = e_4 = \frac{n_4 - 1}{4} \cdot \frac{n}{n_4}.$$

If n is odd, then $n_4 = n$; if n is even, then (since n is not divisible by 4), $n_4 = \frac{n}{2}$. In either case, we get

$$\lambda_{3,1}(\mathbb{Z}_n) = e_4 = \frac{n_4 - 1}{4} \cdot \frac{n}{n_4} = \left\lfloor \frac{n}{4} \right\rfloor.$$

The claims of Theorem 6 now readily follow. $\square$

5 Further comments and open questions

In this final section, we discuss some interesting open questions.

Our first question is about a possible generalization of Theorems 1 and 6. Note that, according to Theorem 3, we have

$$\lambda_{k,l}(\mathbb{Z}_n) \leq \max_{d|n} \left\{ \left(\left\lfloor \frac{d-2}{k+l} \right\rfloor + 1 \right) \cdot \frac{n}{d} \right\}.$$

Question 1 *Let $D(n)$ be the set of divisors of n (which are greater than 1). Given distinct positive integers k and l, is there a subset $D_{k,l}(n)$ of $D(n)$ so that*

$$\lambda_{k,l}(\mathbb{Z}_n) = \max_{d \in D_{k,l}(n)} \left\{ \left(\left\lfloor \frac{d-2}{k+l} \right\rfloor + 1 \right) \cdot \frac{n}{d} \right\}?$$

As we see from (2), Question 1 holds with $D_{k,l}(n) = D(n)$ when n and $k-l$ are relatively prime, in particular, for sum-free sets. According to Theorem 6, the set

$$D_{3,1}(n) = \{d \in D(n) | d \not\equiv 2 \pmod 4\}$$

works for $(k,l) = (3,1)$. (Note that, if it exists, $D_{k,l}(n)$ is not necessarily unique.)

Moving on to general abelian groups, we observe that, by Lemma 10, we have

$$\lambda_{k,l}(G) \geq \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}.$$

Then one of course wonders the following.

Question 2 *Given distinct positive integers k and l, is*

$$\lambda_{k,l}(G) = \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}?$$

According to Theorem 4, Question 2 is equivalent to asking: is

$$\lambda_{k,l}(G) = \max_{d|v} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\}?$$

Note that Theorem 2 of Green and Ruzsa affirms Question 2 for sum-free sets. Theorem 5 exhibits some other cases when the equality also holds. In particular, as a consequence of Theorem 5, we see that

$$\lambda_{3,1}(G) = \lambda_{3,1}(\mathbb{Z}_v) \cdot \frac{n}{v}$$

holds when v (iff n) has at least one prime divisor which is congruent to 3 mod 4, or when v is divisible by 4. So the only cases left open are when $v = P$ or $v = 2P$ where P is the product of primes all of whom are congruent to 1 mod 4.

Next, we are interested in characterizing all (k,l)-sum-free subsets of maximum size.

Question 3 *What are the (k,l)-sum-free subsets A of G with size $|A| = \lambda_{k,l}(G)$?*

A pleasing answer is given by Bier and Chin [5] for the case when $k \geq 3$ and $G \cong \mathbb{Z}_p$ where p is an odd prime: in this case A is an arithmetic progression. The same answer was given by Diananda and Yap [13] earlier for the case when $(k,l) = (2,1)$ (that is, when A is sum-free) and $G \cong \mathbb{Z}_p$ with p not congruent to 1 mod 3; however, for $p = 3m + 1$ the set

$$A = \{m, m+2, m+3, \ldots, 2m-1, 2m+1\}$$

is also sum-free with maximum size. More generally, the answer to Question 3 is known for $(k,l) = (2,1)$ and when n has at least one divisor not congruent to 1 mod 3: in this case A is the union of arithmetic progressions of the same length. More precisely, there is a subgroup H in G so that G/H is cyclic and

$$A = \{(a+H) \cup (a+d+H) \cup \cdots \cup (a+c\cdot d+H)\}$$

for some $a, d \in G$ and integer c. These and other results can be found in [31].

More ambitiously, one may ask for a characterization of all "large" (but not necessarily maximal) (k,l)-sum-free sets in G. Can one, for example, describe explicitly all (k,l)-sum-free sets of size greater than $n/(k+l)$? Hamidoune and Plagne [17] carry this out for sum-free sets of size at least $n/3$ in arbitrary groups. Other results can be found in the papers of Davydov and Tombak [12] and Lev [21], [22].

Our final question is about the number of (k,l)-sum-free subsets in G, which we here denote by $N_{k,l}(G)$.

Question 4 *What is the cardinality $N_{k,l}(G)$ of the set of (k,l)-sum-free subsets in G?*

Clearly, any subset of a (k,l)-sum-free set is also (k,l)-sum-free, so the answer to Question 4 is at least

$$N_{k,l}(G) \geq 2^{\lambda_{k,l}(G)}.$$

But there are indications that the number is not much larger. In fact, for sum-free sets we have the following result of Green and Ruzsa [15]:

$$N_{2,1}(G) = 2^{\lambda_{2,1}(G)+o(1)n},$$

where $o(1)$ approaches zero as n goes to infinity. They have a more accurate approximation for the case when n has a prime divisor which is congruent to 2 mod 3. (This result had been established for even n earlier by Lev, Łuczak, and Schoen [23] and independently by Sapozhenko [28].)

In closing, we mention that the analogues of our questions about the maximum size, the structure, and the number of (k,l)-sum-free sets (especially sum-free sets) have been investigated in non-abelian groups (see Kedlaya's papers [18] and [19]) and, more extensively, among the positive integers (see the works of Alon [1], Bilu [6], Calkin [7], Calkin and Taylor [8], Cameron [9], Cameron and Erdős [10] and [11], and Łuczak and Schoen [24]). General background references on related questions include Nathanson's book [25], Guy's book [16], and Ruzsa's papers [26] and [27]; see also [3] and [4].

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